DIMENSIONAL ENTROPIES AND SEMI-UNIFORM HYPERBOLICITY

JÉRÔME BUZZI

Abstract. We describe *dimensional entropies* introduced in [6], list some of their properties, giving some proofs. These entropies allowed the definition in [7, 10] of *entropy-expanding maps*. We introduce a new notion of *entropy-hyperbolicity* for diffeomorphisms. We indicate some simple sufficient conditions (some of them new) for these properties. We conclude by some work in progress and more questions.

1. Introduction

We are interested in using robust entropy conditions to study chaotic dynamical systems. These entropy conditions imply some "semi-uniform" hyperbolicity. This is a type of hyperbolicity which is definitely weaker than classical uniform hyperbolicity but which is stronger than Pesin hyperbolicity, that is, non vanishing of the Lyapunov exponents of some relevant measure. This type of conditions allows the generalization of some properties of interval maps and surface diffeomorphisms to arbitrary dimensions.

In this paper, we first explain what is known in low dimension just assuming the non-vanishing of the topological entropy $h_{\rm top}(f)$. Then we give a detailed description of the *dimensional entropies*. These are $d+1$ numbers, if d is the dimension of the manifold,

$$0 = h^0_{\rm top}(f) \le h^1_{\rm top}(f) \le \cdots \le h^d_{\rm top}(f) = h_{\rm top}(f).$$

$h^k_{\rm top}(f)$ "counts" the number of orbits starting from an arbitrary compact and smooth k-dimensional submanifold. We both recall known properties and establish new ones. We then recall the definition of *entropy-expanding maps* which generalize the complexity of interval dynamics with non-zero topological entropy. We also introduce a similar notion for diffeomorphisms:

Definition 1. *A diffeomorphism of a d-dimensional manifold is* **entropy-hyperbolic** *if there are integers d_u, d_s such that:*

- $h^{d_u}_{\rm top}(f) = h_{\rm top}(f)$ *and this fails for every dimension $k < d_u$;*
- $h^{d_s}_{\rm top}(f^{-1}) = h_{\rm top}(f)$ *and this fails for every dimension $k < d_s$;*
- $d_u + d_s = d$.

We give simple sufficient conditions for entropy-expansion and entropy-hyperbolicity. Finally we announce some work in progress and state a number of questions.

Date: Talk at ICMP - Rio de Janeiro, Brazil, August 2006.

We now recall some classical notions which may be found in [20].

A basic measure of orbit complexity of a map $f : M \to M$ is the *entropy*. The *topological entropy* $h_{\text{top}}(f)$ "counts" all the orbits and the *measure-theoretic entropy* (also known as Kolmogorov-Sinai entropy or ergodic entropy) $h(f, \mu)$ "counts" the orbits "relevant" to some given invariant probability measure μ. They are related by the following rather general variational principle. If, e.g., f is continuous and M is compact, then

$$h_{\text{top}}(f) = \sup_{\mu} h(f, \mu)$$

where μ ranges over all invariant probability measures. One can also restrict μ to *ergodic* invariant probability measures.

This brings to the fore measures which realize the above supremum, when they exist, and more generally measures which have entropy close to this supremum.

As $\mu \mapsto h(f, \mu)$ is affine, μ has maximum entropy if and only if almost every ergodic component of it has maximum entropy. Hence, with respect to entropy, it is enough to study ergodic measures.

Definition 2. *A* **maximum measure** *is an ergodic and invariant probability measure μ such that $h(f, \mu) = \sup_{\nu} h(f, \nu)$.*

A **large entropy measure** *is an ergodic and invariant probability measure μ such that $h(f, \mu)$ is close to* $\sup_{\nu} h(f, \nu)$.

The *Lyapunov exponents* for some ergodic and invariant probability measure μ are the possible values μ-a.e. of the limit $\lim_{n\to\infty} \frac{1}{n} \log \|T_x f^n.v\|$ where $\|\cdot\|$ is some Riemannian structure and $T_x f$ is the differential of f and v ranges over the non-zero vectors of the tangent space $T_x M$.

A basic result connecting entropy and hyperbolicity is the following theorem (proved by Margulis for volume preserving flows):

Theorem 1 (Ruelle's inequality)**.** *Let $f : M \to M$ be a C^1 map on a compact manifold. Let μ be an f-invariant ergodic probability measure. Let $\lambda_1(\mu) \geq \dots$ be its Lyapunov exponents repeated according to multiplicity. Then,*

$$h(f, \mu) \leq \sum_{i=1}^{d} \lambda_i(\mu)^+$$

In good cases (with enough hyperbolicity), the entropy is also reflected in the existence of many periodic orbits:

Definition 3. *The periodic points of some map $f : M \to M$ satisfy a* **multiplicative lower bound***, if, for some integer $p \geq 1$:*

$$\liminf_{n\to\infty, p|n} e^{-nh_{\text{top}}(f)} \#\{x \in [0,1] : f^n x = x\} > 0.$$

Recall that many diffeomorphisms have infinitely many more periodic orbits (see [17, 18]).

The following type of isomorphism will be relevant to describe all "large entropy measures".

Definition 4. *For a given measurable dynamical system $f : M \to M$, a subset $S \subset M$ is* **entropy-negligible** *if there exists $h < \sup_{\mu} h(f, \mu)$ such that for all ergodic and invariant probability measures μ with $h(f, \mu) > h$, $\mu(S) = 0$.*

An **entropy-conjugacy** *between two measurable dynamical systems $f: M \to M$ and $g: N \to N$ is a bi-measurable invertible mapping $\psi: M \setminus M_0 \to N \setminus N_0$ such that: ψ is a conjugacy (i.e., $g \circ \psi = \psi \circ f$) and M_0 and N_0 are entropy-negligible.*

2. Low Dimension

Low dimension dynamical systems here means interval maps and surface diffeomorphisms - those systems for which non-zero entropy is enough to ensure hyperbolicity of the large entropy measures.

2.1. **Interval Maps.** Indeed, an immediate consequence of Ruelle's inequality on the interval is that a lower bound on the measure-theoretic entropy gives a lower bound on the (unique) Lyapunov exponent. Thus, invariant measures with nonzero topological entropy are hyperbolic in the sense of Pesin. One can obtain much more from the topological entropy:

Theorem 2. *Let $f : [0, 1] \to [0, 1]$ be C^∞. If $h_{\mathrm{top}}(f) > 0$ then f has finitely many maximum measures. Also the periodic points satisfy a multiplicative lower bound.*

This was first proved by F. Hofbauer [15, 16] for piecewise monotone maps (admitting finitely many points $a_0 = 0 < a_1 < \cdots < a_N$ such that $f|]a_i, a_{i+1}[$ is continuous and monotone). It was then extended to arbitrary C^∞ maps in [5]. In both settings, one builds an entropy-conjugacy to a combinatorial model called a Markov shift (which is a subshift of finite type over an infinite alphabet). One can then apply some results of D. Vere-Jones [26] and B. Gurevič [14].

We can even classify these dynamics. Recall that the *natural extension* of $f : M \to M$ is $\bar{f} : \bar{M} \to \bar{M}$ defined as $\bar{M} := \{(x_n)_{n\in\mathbb{Z}} \in M^{\mathbb{Z}} : \forall n \in \mathbb{Z}\ x_{n+1} = f(x_n)\}$ and $\bar{f}((x_n)_{n\in\mathbb{Z}}) = (f(x_n))_{n\in\mathbb{Z}}$. Recall that $\bar{\pi} : (x_n)_{n\in\mathbb{Z}} \mapsto x_0$ induces a homeomorphism between the spaces of invariant probability measures which respects entropy and ergodicity.

Theorem 3. *The natural extensions of C^∞ interval maps with non-zero topological entropy are classified up to entropy-conjugacy by their topological entropy and finitely many integers (which are "periods" of the maximum measures).*

The classification is deduced from the proof of the previous theorem by using a classification result [2] for the invertible Markov shifts involved.

The C^∞ is necessary: for each finite r, there are C^r interval maps with non-zero topological entropy having infinitely many maximum measures and others with none.

Remark 5. *These examples show in particular that Pesin hyperbolicity of maximum measures or even of large entropy measures (which are both consequences of Ruelle's inequality here) are not enough to ensure the finite number of maximum measures.*

2.2. **Surface Transformations.** As observed by Katok [19], Ruelle's inequality applied to a surface diffeomorphism and its inverse (which has opposite Lyapunov exponents) shows that a lower-bound on measure-theoretic entropy bounds away from zero the Lyapunov exponents of the measure. Thus, for surface diffeomorphisms also, nonzero entropy implies Pesin hyperbolicity.

It is believed that surface diffeomorphisms should behave as interval maps, leading to the following folklore conjecture:

Conjecture 1. *Let $f : M \to M$ be a C^∞ surface diffeomorphism. If $h_{\mathrm{top}}(f) > 0$ then f has finitely many maximum measures.*

I would think that, again like for interval maps, finite smoothness is not enough for the above result. However counter-examples to this (or to existence) are known only in dimension ≥ 4 [22].

The best result for surface diffeomorphisms at this point is the following "approximation in entropy" [19]:

Theorem 4 (A. Katok)**.** *Let $f : M \to M$ be a $C^{1+\epsilon}$ surface diffeomorphism. For any $\epsilon > 0$, there exists a horseshoe*[1] *$\Lambda \subset M$ such that $h_{\mathrm{top}}(f|\Lambda) > h_{\mathrm{top}}(f) - \epsilon$. In particular, the periodic points of f satisfy a logarithmic lower bound:*

$$\limsup_{n\to\infty} \frac{1}{n} \log \#\{x \in M : f^n(x) = x\} \geq h_{\mathrm{top}}(f).$$

Katok in fact proved a more general fact, valid for any $C^{1+\epsilon}$-diffeomorphism of a compact manifold of any dimension. Namely, if μ is an ergodic invariant probability measure without zero Lyapunov exponent :

$$\limsup_{n\to\infty} \frac{1}{n} \log \#\{x \in M : f^n(x) = x\} \geq h(f, \mu).$$

On surfaces, Ruelle's inequality and the variational principle imply the theorem as explained above.

I have proved the conjecture for a model class, which replaces distortion with (simple) singularities [8]:

Theorem 5. *Let $f : M \to M$ be a piecewise affine homeomorphism. If $h_{\mathrm{top}}(f) > 0$ then f has finitely many maximum measures.*

3. Dimensional Entropies

We are going to define the dimensional entropies for a smooth self-map or diffeomorphism $f : M \to M$ of a d-dimensional compact manifold. We will then investigate these quantities by considering other growth rates obtained from the volume and size of the derivatives. Finally we shall establish the topological variational principle stated in the introduction by a variant of Pliss Lemma.

3.1. **Singular disks.** The basic object is:

Definition 6. *A* **(singular)** *k-***disk** *is a map $\phi : Q^k \to M$ with $Q^k := [-1, 1]^k$. It is C^r if it can be extended to a C^r map on a neighborhood of Q^k.*

We need to define the C^r size $\|\phi\|_{C^r}$ of a singular disk ϕ for $1 \leq r \leq \infty$ as well as the corresponding topologies on the space of such disks. This involves some technicalities as vectors in different tangent spaces are not comparable *a priori*. We refer to Appendix A for the precise definitions, which are rather obvious for finite r. For $r = \infty$, we need an approximation property (which fails for some otherwise very reasonable definitions of C^r size), Fact 40, which is used to prove Lemma 10 below.

From now on, we fix some C^r size arbitrarily on the manifold M. We will later check that the entropies we are interested in are in fact independent of this choice.

[1] A horseshoe is an invariant compact subset on which some iterate of f is conjugate with a full shift on finitely many symbols.

Notations. It will be convenient to sometimes write ϕ instead of $\phi(Q^k)$, e.g., $h_{\rm top}(f,\phi)$ instead of $h_{\rm top}(f,\phi(Q^k))$.

3.2. **Entropy of collections of subsets.** Given a collection $\mathcal{D}$ of subsets of M, we associate the following entropies. Recall that the (ϵ,n)-covering number of some subset $S\subset M$ is:

$$r_f(\epsilon,n,S):=\min\{\#C:\bigcup_{x\in S}B_f(\epsilon,n,x)\supset S\}$$

where $B_f(\epsilon,n,x):=\{y\in M:\forall 0\le k<n\ d(f^ky,f^kx)<\epsilon\}$ is the (ϵ,n)-dynamic ball. The classical Bowen-Dinaburg formula for the topological entropy of $S\subset M$ is $h_{\rm top}(f,S)=\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log r_f(\epsilon,n,S)$ and $h_{\rm top}(f)=h_{\rm top}(f,M)$.

Definition 7. *The* **topological entropy** *of $\mathcal{D}$ is:*

$$h_{\rm top}(f,\mathcal{D}):=\sup_{D\in\mathcal{D}}h_{\rm top}(f,D)=\sup_{D\in\mathcal{D}}\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log r_f(\epsilon,n,D)$$

The **uniform topological entropy** *of $\mathcal{D}$ is:*

$$H_{\rm top}(f,\mathcal{D}):=\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\sup_{D\in\mathcal{D}}r_f(\epsilon,n,D).$$

Clearly $h_{\rm top}(f,\mathcal{D})\le H_{\rm top}(f,\mathcal{D})$. The inequality can be strict as shown in the following examples (the first one involving non-compactness, the second one involving non-smoothness).

Example 1. *Let $T:\mathbb{T}^2\to\mathbb{T}^2$ be a linear endomorphism with two eigenvalues Λ_1,Λ_2 with $1<|\Lambda_1|<|\Lambda_2|$. Let $\mathcal{L}$ be the set of finite line segments. We have*

$$0<h_{\rm top}(T,\mathcal{L})=\log|\Lambda_2|<H_{\rm top}(T,\mathcal{L})=\log|\Lambda_1|+\log|\Lambda_2|.$$

Example 2. *There exist a C^∞ self-map F of $[0,1]^2$ and a collection $\mathcal{C}$ of C^r curves with bounded C^r norm such that $0<h_{\rm top}(F,\mathcal{C})<H_{\rm top}(F,\mathcal{C})$. This can be deduced from the example with $h^1_{\rm top}(f\times g)>\max(h_{\rm top}(f),h_{\rm top}(g))$ in* [6] *by considering curves with finitely many bumps converging C^{r-1} to the example curve there, which has infinitely many bumps.*

3.3. **Definitions of the dimensional entropies.** We can now properly define the dimensional entropies. Recall that we have endowed M with a C^r size.

Definition 8. *For each $1\le r\le\infty$, the* **standard family** *of C^r singular k-disks is the collection of all C^r singular k-disks. For finite r, the* **standard uniform family** *of C^r singular k-disks is the collection of all C^r singular k-disks with C^r size bounded by 1.*

Definition 9. *The C^r,* **k-dimensional entropy** *of a self-map f of a compact manifold is:*

$$h^{k,C^r}_{\rm top}(f):=h_{\rm top}(f,\mathcal{D}^k_r)$$

where $\mathcal{D}^k_r$ is a standard family of C^r k-disks of M. We write $h^k_{\rm top}(f)$ for $h^{k,C^\infty}(f)_{\rm top}$ and call it the k-dimensional entropy.

The C^r, **k-dimensional uniform entropy** *$H^{k,C^r}_{\rm top}(f)$ is obtained by replacing $h_{\rm top}(f,\mathcal{D}^k_r)$ with $H_{\rm top}(f,D^k_r)$ in the above definition where D^k_r is the standard uniform family. We write $H^k_{\rm top}(f)$ for $H^{k,C^\infty}_{\rm top}(f)$ and call it the k-dimensional uniform entropy.*

Observe that $h_{\rm top}^{k,C^r}(f)$ and $H_{\rm top}^{k,C^r}(f)$ are non-decreasing functions of k and non-increasing functions of r. Indeed, (1) $\mathcal{D}_r^k \supset \mathcal{D}_s^k$ and $D_r^k \supset D_s^k$ if $r \leq s$; (2) for any $0 \leq \ell \leq k \leq d$, restricting a k-disk to $[0,1]^\ell \times \{0\}^{k-\ell}$ does not increase its C^r size. Observe also that $h_{\rm top}^{0,C^r}(f) = 0$ and $h_{\rm top}^d(f) = h_{\rm top}(f)$.

Lemma 10. *Let $f : M \to M$ be a C^∞ self-map of a compact manifold. We have:*

$$H_{\rm top}^k(f) = \lim_{r\to\infty} H_{\rm top}^{k,C^r}(f)$$

and the limit is non-increasing.

We shall see later in Proposition 26 that the same holds for $h_{\rm top}^k(f)$.

Proof. We use one of the (simpler) ideas of Yomdin's theory. For each $n \geq 1$, we divide Q^k into small cubes with diameter at most $(\epsilon/4)^{1/r}\|\phi\|_{C^r}^{-1/r}\mathrm{Lip}(f)^{-n/r}$. We need $(\epsilon/4)^{-k/r}\sqrt{k}^k\|\phi\|_{C^r}^{k/r}\mathrm{Lip}(f)^{\frac{k}{r}n}$ such cubes. Let q be one of them. By Fact 40, there exists a C^∞ k-disk ϕ_q such that $\|\phi_q\|_{C^\infty} \leq 2\|\phi\|_{C^r}$ and

$$\forall t \in q \quad d(\phi_q(t),\phi(t)) \leq \|\phi\|_{C^r}\|t-t_q\|^r \leq \|\phi\|_{C^r} \times \frac{\epsilon}{2}\|\phi\|_{C^r}^{-1}\mathrm{Lip}(f)^{-n} \leq \frac{\epsilon}{2}\mathrm{Lip}(f)^{-n}$$

It follows that $r_f(\epsilon,n,\phi\cap q) \leq r_f(\epsilon/2,n,\phi_q)$. Thus,

$$r_f(\epsilon,n,\mathcal{D}_r^k) \leq \sqrt{k}^k(\epsilon/4)^{-k/r}\|\phi\|_{C^r}^{k/r}\mathrm{Lip}(f)^{\frac{k}{r}n}r_f(\epsilon/2,n,\mathcal{D}_\infty^k)$$

Hence, writing $\mathrm{lip}(f) := \max(\log\mathrm{Lip}(f),0)$,

$$H_{\rm top}^{k,C^r}(f) \leq \frac{k}{r}\mathrm{lip}(f) + H_{\rm top}^k(f).$$

The inequality $H_{\rm top}^{k,C^r}(f) \geq H_{\rm top}^k(f)$ is obvious, concluding the proof. $\square$

Lemma 11. *The numbers $H_{\rm top}^{k,C^r}(f)$ do not depend on the underlying choice of a C^r size.*

Proof. Using Lemma 10, it is enough to treat the case with finite smoothness. Let $\mathcal{D}_1,\mathcal{D}_2$ be two standard families of k-disks, defined by two C^r sizes $\|\cdot\|_{C^r}^1, \|\cdot\|_{C^r}^2$. By Fact 39, there exists $C < \infty$ such that $\|\cdot\|_{C^r}^1 \leq C\|\cdot\|_{C^r}^2$. Hence setting $K := ([C]+1)^k$, for any k-disk $\phi_1 \in \mathcal{D}_1$ can be *linearly subdivided*[2] into K k-disks $\phi_2^1,\dots,\phi_2^K \in \mathcal{D}_2$. Thus

$$\forall n \geq 0 \quad r_f(\epsilon,n,\phi_1) \leq K\max_j r_f(\epsilon,n,\phi_2^j).$$

It follows immediately that $H(f,\mathcal{D}_1) \leq H(f,\mathcal{D}_2)$. The claimed equality follow in turn by symmetry. $\square$

4. Other growth rates of submanifolds

4.1. **Volume growth.** Entropy is a growth rate under iteration. Equipping M with a Riemannian structure allows the definition of volume growth of submanifolds.

[2]That is, each $\phi_2^j = \phi_1 \circ L_j$ with $L_j : Q^k \to Q^k$ linear and $\bigcup_{j=1}^K L_j(Q^k) = Q^k$.

Definition 12. *Let $\phi : Q^k \to M$ be a singular k-disk. Its* **(upper) growth rate** *is:*

$$\gamma(f,\phi) := \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{vol}(f^n \circ \phi) \text{ with } \operatorname{vol}(\psi) := \int_{Q^k} \|\Lambda^k \psi(x)\|\, dx$$

where $\|\Lambda^k\psi\|$ is the Jacobian of $\psi : Q^k \to M$ wrt the obvious Riemannian structures. The volume growth exponent of f in dimension k is:

$$\gamma^k(f) := \sup_{\phi\in\mathcal{D}^k_r} \gamma(f,\phi),$$

$\gamma(f) := \max_{0\le k\le d} \gamma^k(f)$ is simply called the **volume growth** *of f.*

Observe that the value of the growth rates defined above are independent of the choice of the Riemannian structure, by compactness of the manifold.

The volume growth dominates the entropy quite generally:

Theorem 6 (Newhouse [23])**.** *Let $f : M \to M$ be a $C^{1+\alpha}$, $\alpha > 0$, smooth self-map of a compact manifold. Then:*

$$h_{\text{top}}(f) \le \gamma(f).$$

Remark 13. *More precisely, his proof gave*

$$h_{\text{top}}(f) \le \gamma^{d^{cu}}(f)$$

where d^{cu} is such that the variational principle $h_{\text{top}}(f) = \sup_\mu h(f,\mu)$ still holds when μ is restricted to measures with exactly d^{cu} nonpositive Lyapunov exponents.

For C^{1+1}-diffeomorphisms, deeper ergodic techniques due to Ledrappier and Young are available and Cogswell [11] *has shown that, for any ergodic invariant probability measure μ, there exists a disk Δ such that $h_{\text{top}}(f,\mu) \le h_{\text{top}}(f,\Delta) \le \gamma(f,\Delta)$. More precisely, the dimension of this disk is the number of positive Lyapunov exponents. For C^∞ diffeomorphisms (more generally if there is a maximum measure) there exists a disk $\Delta_{\max}$ such that*

$$h_{\text{top}}(f) = h_{\text{top}}(f,\Delta_{\max}) = \gamma(f,\Delta_{\max}).$$

I do not know if Newhouse's inequality fails for C^1 maps.

The proof of Newhouse inequality involves *ergodic theory* and especially *Pesin theory*. Indeed, this type of inequality does not hold uniformly:

Example 3. *There exist a C^∞ self-map F of a surface and a C^∞ curve ϕ such that, for some sequence $n_i \to \infty$,*

$$\lim_{i\to\infty} \frac{1}{n_i} \log r_F(\epsilon, n_i, \phi) > \lim_{i\to\infty} \frac{1}{n_i} \log \operatorname{vol}(F^{n_i} \circ \phi).$$

Proof. Let $\alpha > 0$ be some small number. Let $I := [0,1]$. Let $f : I \to I$ be a C^∞ map such that: (i) $f(0) = f(1) = 0$; (ii) $f(1/2) = 1$; (iii) $f|[0,1/2]$ is increasing and $f|[1/2,1]$ is decreasing; (iv) $f'|[0,1/2-\alpha] = 2(1+\alpha)$ and $f'|[1/2+\alpha] = -2(1+\alpha)$. As α is small, $1/2$ has a preimage in $[0,1/2-\alpha]$. Let x_{-n} be the leftmost preimage in $f^{-n}(1)$: $x_0 = 1$, $x_{-1} = 1/2$, and for all $n \ge 2$, $x_{-n} = 2^{-n}(1+\alpha)^{-n+1}$. Let $g : I \to I$ be another C^∞ map such that: (i) $g(0) = 0$; (ii) $0 < g' < 1$; (iii) $g(x_{-n}) = x_{-n-1}$ for all $n \ge 0$.

Consider the following composition of length $3n$ for some $n \ge 1$:

$$I \longrightarrow^{f^n} 2^n \times I \longrightarrow^{g^n} 2^n \times [0,x_{-n}] \longrightarrow^{f^n} 2^n \times I. \tag{1}$$

Observe that after time $2n$, the length of the curve $g^n \circ f^n$ is $2^n \cdot x_{-n} = (1+\alpha)^{-n+1}$ whereas the number of (ϵ, n)-separated orbits is less than $\epsilon^{-1} n 2^n$. After time $3n$, the curve $f^n \circ g^n \circ f^n$ has image I with multiplicity 2^n. It is therefore easy to analyze the dynamics of compositions of such sequences.

We build our example by considering a skew-product for which the curve will be a fiber over a point which will drive the application of sequences as above.

Let $h : S^1 \to S^1$ be the circle map defined by $h(\theta) = 4\theta \mod 2\pi$. Let $F : S^1 \times I \to S^1 \times I$ be a C^∞ map such that: $F(\theta, x) = (h(\theta), f(x))$ if $\theta \in [0, \frac{1}{6}]$ and $F(\theta, y) = (h(\theta), g(x))$ if $\theta \in [\frac{1}{2}, \frac{2}{3}]$

Recall that the expansion in basis 4 of $\theta \in S^1$ is the sequence $a_1 a_2 \cdots \in \{0,1,2,3\}^{\mathbb{N}}$ such that $\theta = 2\pi \sum_{k \geq 1} a_k 4^{-k}$. We write $\theta = \overline{0.a_1 a_2 a_3 \dots}^4$.

Observe that, whenever θ has only 0s and 2s in its expansion,

$$h^n(\theta) \in [0, \frac{1}{6}] = [0, \overline{0.02222\dots}^4]$$

whenever its nth digit is 0 and

$$h^n(\theta) \in [\frac{1}{2}, \frac{2}{3}] = [\overline{0.2000\dots}^4, \overline{0.222\dots}^4]$$

whenever its nth digit is 2. Thus we can specify the desired compositions of f and g just by picking $\theta \in S^1$ with the right expansion. We pick:

$$\theta_1 = \overline{0.0^{n_1} 2^{n_1} 0^{n_1+n_2} 2^{n_2} 0^{n_2+n_3} 2^{n_3} 0^{n_3+n_4} \dots}^4$$

so that we shall have a sequence of compositions of the type (1). We write $N_i := 3n_1 + \dots + 3n_i$. We set $n_i := i!$ so that $n_{i+1}/N_i \to \infty$. Let $\phi^1 : Q^1 \to S^1 \times I$ be defined by $\phi^1(s) = (\theta_1, (s+1)/2)$.

The previous analysis shows that $F^{N_i} \circ \phi^1$ has image I with multiplicity $2^{n_1 + \dots + n_i} = 2^{\frac{1}{3} N_i}$. $F^{N_i + n_{i+1}} \circ \phi^1$ has image I with multiplicity $2^{\frac{1}{3} N_i} \times 2^{n_{i+1}}$. $F^{N_i + n_{i+1}} \circ \phi^1$ has image $[0, x_{-n_{i+1}}]$ with multiplicity $2^{\frac{2}{3} N_i} \times 2^{n_{i+1}}$. It follows that, setting $t_i := N_i + 2n_{i+1} \approx 2n_{i+1}$,

$$\log r_F(\epsilon, t_i, \phi^1) \approx (\frac{1}{3} N_i + n_{i+1}) \log 2$$

whereas

$$\operatorname{vol}(F^{t_i} \circ \phi^1) = x_{-n_{i+1}} \times 2^{\frac{1}{3} N_i + n_{i+1}} = (1+\alpha)^{-n_{i+1}} 2^{\frac{1}{3} N_i}.$$

Hence,

$$\frac{1}{t_i} \log r_F(\epsilon, t_i, \phi^1) \approx \frac{1}{2} \log 2 \text{ whereas } \frac{1}{t_i} \log \operatorname{vol}(F^{t_i} \circ \phi^1) \preceq -\frac{1}{2} \alpha,$$

as claimed. □

Remark 14. *The inequality in the previous example is obtained as the length is contracted after a large expansion. For curves, this is in fact general and it is easily shown that, for any C^1 1-disk ϕ with unit length, for any $0 < \epsilon < 1$:*

$$\forall n \geq 0 \quad \epsilon \cdot r_f(\epsilon, n, \phi) \leq \max_{0 \leq k < n} \operatorname{vol}(f^k \circ \phi) + 1. \tag{2}$$

(2) implies that, for curves,

$$h_{\text{top}}(f, \phi) \leq \gamma(f, \phi), \tag{3}$$

both quantities being defined by $\limsup$ *(this would fail using* $\liminf$*). However, one can find similarly as above, a* C^∞ *self-map of a* 3*-dimensional compact manifold and a* C^∞ *smooth* 2*-disk such that (2) fails though (3) seems to hold.*

We ask the following:

Question. *Let* $f : M \to M$ *be a* C^∞ *self-map of a compact* d*-dimensional manifold. Is it true that, for any singular* k*-disk* ψ $(0 \le k \le d)$

$$h_{\mathrm{top}}(f, \psi) \le \max_{\phi \subset \psi} \gamma(f, \phi) \quad ?$$

(both rates being defined using $\limsup$ *and* ϕ *ranging over singular* ℓ*-disks,* $0 \le \ell < k$*, with* $\phi(Q^\ell) \subset \psi(Q^k)$*)? Is it at least true that*

$$h^k_{\mathrm{top}}(f) \le \max_{0 \le \ell \le k} \gamma^\ell(f) \quad ?$$

These might even hold for finite smoothness for all I know.

Conversely, entropy also provides some bounds on volume growth

Theorem 7 (Yomdin [27])**.** *Let* $f : M \to M$ *be a* C^r*,* $r \ge 1$*, smooth self-map of a compact manifold. Let* $\alpha > 0$*. Then there exist* $C(r, \alpha) < \infty$ *and* $\epsilon_0(r) > 0$ *with the following property. Let* $\phi : Q^k \to M$ *be any* C^r *singular* k*-disk with unit* C^r *size for some* $0 \le k \le d$*. Then, for any* $n \ge 0$*,*

$$\mathrm{vol}(f^n \circ \phi) \le C(r, \alpha)\mathrm{Lip}(f)^{(\frac{k}{r}+\alpha)n} \cdot r_f(\epsilon_0, n, \phi).$$

In particular,

$$\gamma^k(f) \le h^k_{\mathrm{top}}(f) + \frac{k}{r}\mathrm{lip}(f).$$

Remark 15. *The above extra term is indeed necessary as shown already by examples attributed by Yomdin* [27] *to Margulis: there is* $f : [0, 1] \to [0, 1]$*,* C^r *with* $h_{\mathrm{top}}(f) = 0$ *and* $\gamma(f) = \mathrm{lip}(f)/r$*.*

Remark 16. *Yomdin's estimate is* uniform *holding for each disk and each iterate. Its proof involves very little dynamics and no ergodic theory, in contrast to Newhouse's inequality quoted above.*

Corollary 17. *Let* $f : M \to M$ *be a self-map of a compact manifold. If* f *is* C^∞*, then*

$$h_{\mathrm{top}}(f) = \gamma(f).$$

Let $f_* : H_*(M, \mathbb{R}) \to H_*(M, \mathbb{R})$ be the total homological action of f. Let $\rho(f_*)$ be its spectral radius. As the ℓ^1-norm in homology gives a lower bound on the volume, we have $\gamma(f) \ge \log \rho(f_*)$. Hence, the following special case of the Shub Entropy Conjecture is proved:

Corollary 18 (Yomdin [27])**.** *Let* $f : M \to M$ *be a self-map of a compact manifold. If* f *is* C^∞*, then*

$$\log \rho(f_*) \le h_{\mathrm{top}}(f).$$

4.2. **Resolution entropies.** The previous results of Yomdin and more can be obtained by computing a growth rate taking into account the full structure of singular disks. A variant of this idea is explained in Gromov's Bourbaki Seminar [12] on Yomdin's results. We build on [4].

Definition 19. *Let $r \geq 1$. Let $\phi : Q^k \to M$ be a C^r singular k-disk. A* **C^r-resolution** *$\mathcal{R}$ of order n of ϕ is a collection of C^r maps $\psi_\omega : Q^k \to Q^k$, for $\omega \in \Omega$ with Ω a finite collection of words of length at most n with the following properties. For each $\omega \in \Omega$, let $\Psi_\omega := \psi_{\sigma^{|\omega|-1}\omega} \circ \cdots \circ \psi_\omega(Q^k)$. We require:*

(1) $\bigcup_{|\omega|=n} \Psi_\omega(Q^k) = Q^k$;
(2) $\|\psi_\omega\|_{C^r} \leq 1$ *for all* $\omega \in \Omega$;
(3) $\|f^{|\omega|} \circ \Psi_\omega\|_{C^r} \leq 1$ *for all* $\omega \in \Omega$,

The **size** *$|\mathcal{R}|$ of the resolution is the number of words in Ω with length n.*

Condition (2) added in [4] much simplifies the link between resolutions and entropy. It no longer relies on Newhouse application of Pesin theory and becomes straightforward:

Fact 20. *Let $\mathcal{R} := \{\psi_\omega : Q^k \to Q^k : \omega \in \Omega\}$ be a C^r-resolution of order n of $\phi : Q^k \to M$. Let $\epsilon > 0$ and Q^k_ϵ be ϵ-dense in Q^k, i.e., $Q^k \subset \bigcup_{t \in Q^k_\epsilon} B(x, \epsilon)$. Then*

$$\{\Psi_\omega(t) : t \in Q^k_\epsilon \text{ and } \omega \in \Omega \text{ with } |\omega| = n\} \text{ is a } (\epsilon, n)\text{-cover of } \phi(Q^k).$$

On the other hand, the notion of resolution induces entropy-like quantities:

Definition 21. *Let $1 \leq r < \infty$ and let $f : M \to M$ be a C^r self-map of a compact manifold. Let $R_f(C^r, n, \phi)$ be the minimal size of a C^r-resolution of order n of a C^r singular disk ϕ. The* **resolution entropy** *of ϕ is:*

$$h_R(f, \phi) := \limsup_{n\to\infty} \frac{1}{n} \log R_f(C^r, n, \phi).$$

If $\mathcal{D}$ is a collection of C^r singular disks, its C^r resolution entropy is

$$h_{R,C^r}(f, \mathcal{D}) := \sup_{\phi \in \mathcal{D}} h_R(f, \phi)$$

and its C^r uniform resolution entropy is:

$$H_{R,C^r}(f, \mathcal{D}) := \limsup_{n\to\infty} \frac{1}{n} \log R_f(C^r, n, \mathcal{D})$$

where $R_f(C^r, n, \mathcal{D}) := \max_{\phi \in \mathcal{D}} R_f(C^r, n, \phi)$. We set:

$$h_R^{k,C^r}(f) = h_{R,C^r}(f, \mathcal{D}^k_r) \text{ and } H_R^{k,C^r}(f) = H_{R,C^r}(f, D^k_r).$$

The following is immediate but very important:

Fact 22. *Let $1 \leq r < \infty$ and $0 \leq k \leq d$. Let $f : M \to M$ be a C^r self-map of a compact d-dimensional manifold. The sequence $n \mapsto R_f(C^r, n, D^k_r)$ is sub-multiplicative:*

$$R_f(C^r, n+m, D^k_r) \leq R_f(C^r, n, D^k_r) R_f(C^r, m, D^k_r).$$

The key technical result of Yomdin's theory can be formulated as follows:

Proposition 23. *Let $1 \leq r < \infty$ and $\alpha > 0$. Let $f : M \to M$ be a C^r self-map of a compact manifold. There exist constants $C', C(r,\alpha), \epsilon_0(r,\alpha)$ with the following property. For any C^r singular disk ϕ, any number $0 < \epsilon < \epsilon_0(r,\alpha)$ and any integer $n \geq 1$,*

$$C'\epsilon^k r_f(\epsilon, n, \phi) \leq R_f(C^r, n, \phi) \leq C(r,\alpha)\mathrm{Lip}(f)^{(\frac{k}{r}+\alpha)n} r_f(\epsilon, n, \phi).$$

Remark that the above constants depend on f. The first inequality follows from Fact 20. The second is the core of Yomdin theory, we refer to [4] for details.

5. Properties of Dimensional Entropies

We turn to various properties of dimensional entropies, most of which can be shown using resolution entropy and its submultiplicativity.

5.1. **Link between Topological and Resolution Entropies.**

We start by observing that Proposition 23 links the topological and resolution entropies.

Corollary 24. *For all positive integers r, k, any collection of C^r k-disks $\mathcal{D}$ and any C^r self-map f on a manifold equipped with a C^r size:*

$$h_{\mathrm{top}}(f, \mathcal{D}) \leq h_{R,C^r}(f, \mathcal{D}) \leq h_{\mathrm{top}}(f, \mathcal{D}) + \frac{k}{r}\log \mathrm{Lip}(f)$$

$$H_{\mathrm{top}}(f, \mathcal{D}) \leq H_{R,C^r}(f, \mathcal{D}) \leq H_{\mathrm{top}}(f, \mathcal{D}) + \frac{k}{r}\log \mathrm{Lip}(f).$$

If the disks in $\mathcal{D}$ are C^∞, then, for $r \leq s < \infty$,

$$h_{R,C^r}(f, \mathcal{D}) \leq h_{R,C^s}(f, \mathcal{D}) \leq h_{\mathrm{top}}(f, \mathcal{D}) + \frac{k}{s}\log \mathrm{Lip}(f).$$

Letting $s \to \infty$, we get:

Corollary 25. *If f is C^∞, then, for all $1 \leq r < \infty$,*

$$h_{R,C^r}(f, \mathcal{D}^k_\infty) = h_{\mathrm{top}}(f, \mathcal{D}^k_\infty).$$

The same holds for uniform topological entropy.

5.2. **Gap between Uniform and Ordinary Dimensional Entropies.**

Yomdin theory gives the following relation:

Proposition 26. *Let $1 \leq r < \infty$ and $f : M \to M$ be a C^r self-map of a compact d-dimensional manifold. For each $0 \leq k \leq d$,*

$$h^{k,C^r}_{\mathrm{top}}(f) \leq H^{k,C^r}_{\mathrm{top}}(f) \leq h^{k,C^r}_{\mathrm{top}}(f) + \frac{k}{r}\mathrm{lip}(f)$$

and the same holds for the resolution entropies $h^{k,C^r}_R(f)$ and $H^{k,C^r}_R(f)$.

In particular, in the C^∞ smooth case, all the versions of the dimensional entropies agree: $h^k_{\mathrm{top}}(f) = H^k_{\mathrm{top}}(f) = h^k_R(f) = H^k_R(f) = \lim_{r\to\infty} H^{k,C^r}_{\mathrm{top}}(f)$.

Proof. It is obvious that the uniform entropies dominate ordinary ones. By Fact 20, $h^{k,C^r}_{\mathrm{top}}(f) \leq h^{k,C^r}_R(f)$ and $H^{k,C^r}_{\mathrm{top}}(f) \leq H^{k,C^r}_R(f)$. Therefore it is enough to show:

$$H^{k,C^r}_R(f) \leq h^{k,C^r}_{\mathrm{top}}(f) + \frac{k}{r}\mathrm{lip}(f). \tag{4}$$

Let $\alpha > 0$. Let $\epsilon_0 > 0$ as in Proposition 23. This proposition defines a number $C(r,\alpha)$. By definition, for every $\phi \in D^k_r$, there exists $n_\phi < \infty$ such that

$$r_f(\epsilon_0, n_\phi, \phi) \leq e^{(h^{k,C^r}_{\mathrm{top}}(f)+\alpha)n_\phi}.$$

We can arrange it so that this holds for all k-disks ψ in some C^0 neighborhood $\mathcal{U}_\phi$ of ϕ. We also assume n_ϕ so large that $C(r,\alpha) \le e^{\alpha n_\phi}$.

By Proposition 23, each such ψ admits a resolution with size at most

$$r_f(\epsilon, n_\phi, \psi) \times C(r,\alpha)\mathrm{Lip}(f)^{\frac{k}{r}n_\phi} \le e^{(h_R^{k,C^r}(f)+\frac{k}{r}\mathrm{lip}(f)+2\alpha)n_\phi}.$$

D_r^k is relatively compact in the C^0 topology, hence there is a finite cover $D_r^k \subset \mathcal{U}_{\phi_1} \cup \cdots \cup \mathcal{U}_{\phi_K}$. Let $N := \max n_{\phi_j}$. It is now easy to build, for each $n \ge 0$ and each $\psi \in D_r^k$ a C^r resolution $\mathcal{R}$ of order n with:

$$|\mathcal{R}| \le \exp(h_{\mathrm{top}}^{k,C^r}(f) + \frac{k}{r}\mathrm{lip}(f) + 2\alpha)(n+N).$$

(4) follows by letting α go to zero. □

5.3. **Continuity properties.**

Proposition 27. *We have the following upper semicontinuity properties:*

1. *(1) $f \mapsto H_R^{k,C^r}(f)$ is upper semicontinuous in the C^r topology for all $1 \le r < \infty$;*
2. *(2) $f \mapsto H_{\mathrm{top}}^k(f)$ is upper semicontinuous in the C^∞ topology;*
3. *(3) the defect in upper semi-continuity of $f \mapsto H_{\mathrm{top}}^{k,C^r}(f)$ at $f = f_0$ is at most $\frac{k}{r}\mathrm{lip}(f_0)$:*

$$\limsup_{f\to f_0} H_{\mathrm{top}}^{k,C^r}(f) \le H_{\mathrm{top}}^{k,C^r}(f_0) + \frac{k}{r}\mathrm{lip}(f_0).$$

Proof. We prove (1). The sub-multiplicativity of resolution numbers observed in Fact 22 implies that: $H_R^{k,C^r}(f) = \inf_{n\ge1}\frac{1}{n}\log R_f(C^r,n,D_r^k)$. For each fixed positive integer n, $R_g(C^r,n,D_r^k) \le 2^k R_f(C^r,n,D_r^k)$ for any g C^r-close to f (use a linear subdivision). Thus $f \mapsto H_R^{k,C^r}(f)$ is upper semi-continuous in the C^r topology.

We deduce (3) from (1). Let $f_n \to f$ in the C^r topology. By the preceding, $H_R^{k,C^r}(f) \ge \limsup_{n\to\infty} H_R^{k,C^r}(f_n)$. By Proposition 26, $H_{\mathrm{top}}^k(f) \ge H_R^{k,C^r}(f) - \frac{k}{r}\mathrm{lip}(f)$.

(2) follows from (3) using Lemma 10. □

On the other hand, $f \mapsto H_{\mathrm{top}}^k(f)$ fails to be lower semi-continuous except for interval maps for which topological entropy is lower semi-continuous in the C^0 topology by a result of Misiurewicz. In every case there are counter examples:

Example 4. *For any $d \ge 2$ and $1 \le k \le d$, there is a self-map of a compact manifold of dimension d at which $h_{\mathrm{top}}^k(f)$ fails to be lower semi-continuous.*

Let $h : \mathbb{R} \to [0,1]$ be a C^∞ function such that $h(t) = 1$ if and only if $t = 0$. Let $F_\lambda : [0,1]^d \to [0,1]^d$ be defined by

$$F_\lambda(x_1,\dots,x_d) = (h(\lambda)x_1, 4x_1x_2(1-x_2), x_3,\dots,x_d)$$

Observe that if $\lambda \ne 0$, then $h(\lambda) \in [0,1)$ and $F_\lambda^n(x_1,\dots,x_d)$ approaches $\{(0,0)\} \times [0,1]^{d-2}$ on which F_λ is the identity. Therefore $h_{\mathrm{top}}(F_\lambda) = 0$. On the other hand $h_{\mathrm{top}}(F_0) = h_{\mathrm{top}}(x \mapsto 4x(1-x)) = \log 2$. Now, $H_{\mathrm{top}}^k(F_\lambda) \le h_{\mathrm{top}}(F_\lambda) = 0$ for any $\lambda \ne 0$ and $H_{\mathrm{top}}^k(F_0) \ge h_{\mathrm{top}}^k(f) \ge h_{\mathrm{top}}(F_0, \{1\} \times [0,1] \times \{(0,\dots,0)\}) = \log 2$ for any $k \ge 1$.

6. Hyperbolicity from Entropies

We now explain how the dimensional entropies can yield dynamical consequences. We start by recalling an inequality which will yield hyperbolicity at the level of measures. Then we give the definition and main results for entropy-expanding maps. Finally we explain the new notion of entropy-hyperbolicity for diffeomorphisms.

6.1. A Ruelle-Newhouse type inequality.

One of the key uses of dimensional entropies is to give bounds on the exponents using the following estimate. This will give hyperbolicity of large entropy measure from assumptions on these dimensional entropies.

Theorem 8. [7] *Let $f : M \to M$ be a C^r self-map of a compact manifold with $r > 1$. Let μ be an ergodic, invariant probability measure with Lyapunov exponents $\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_d(\mu)$ repeated according to multiplicity. Recall that $H^k_{\text{top}}(f)$ is the uniform k-dimensional entropy of f. Then:*

$$h(f, \mu) \leq H^k_{\text{top}}(f) + \lambda_{k+1}(\mu)^+ + \cdots + \lambda_d(\mu)^+.$$

Remark 28. *For $k = 0$ this reduces to Ruelle's inequality. For k equal to the number of nonnegative exponents, this is close to Newhouse inequality (with $H^k_{\text{top}}(f)$ replacing $\gamma^k(f)$). The proof is similar to Newhouse's and relies on Pesin theory.*

6.2. Entropy-expanding Maps.

We require that the full topological entropy only appear at the full dimension.

Definition 29. *A C^r self-map $f : M \to M$ of a compact manifold is* **entropy-expanding** *if:*

$$H^{d-1}_{\text{top}}(f) < h_{\text{top}}(f).$$

An immediate class of examples is provided by the interval maps with non-zero topological entropy.

The first consequence of this condition is that ergodic invariant probability measures with entropy $> H^{d-1}_{\text{top}}(f)$ have only Lyapunov exponents bounded away from zero. This follows immediately from Theorem 8.

This also allows the application of (a non-invertible version of) Katok's theorem, proving a logarithmic lower bound on the number of periodic points.

Katok's proof gives horseshoes with topological entropy approaching $h_{\text{top}}(f)$. In particular these maps are points of lower semi-continuity of $f \mapsto h_{\text{top}}(f)$ in any C^r topology, $r \geq 0$. Combining with the upper semi-continuity from Yomdin theory we get:

Proposition 30. *The entropy-expansion property is open in the C^∞ topology.*

Thus we can use the following estimate

Proposition 31. [6] *The Cartesian product of a finite number of C^∞ smooth interval maps, each with nonzero topological entropy is entropy-expanding.*

To get dynamically interesting examples:

Example 5. *For $|\epsilon|$ small enough, the plane map $F_\epsilon : (x, y) \mapsto (1 - 1.8x^2 - \epsilon y^2, 1 - 1.9y^2 - \epsilon x^2)$ preserves $[-1, 1]^2$ and its restriction to this set is entropy-expanding.*

A sufficient condition, considered in a different approach by Oliveira and Viana [24, 25] is the following:

Lemma 32. *Let $f : M \to M$ be a diffeomorphism of a compact Riemanian manifold. Let $\|\Lambda^k Tf\|$ be the maximum over all $1 \leq \ell \leq k$ and all $x \in M$ of the Jacobian of the restrictions of the differential $T_x f$ to any k-dimensional subspace of $T_x M$. Then*

$$H^k_{\text{top}}(f) \leq \log \|\Lambda^k Tf\|.$$

In particular, $\log \|\Lambda^k T\| < h_{\text{top}}(f)$ implies that f is entropy-expanding. An even stronger condition is $(d-1)\text{lip}(f) < h_{\text{top}}(f)$.

The proof of this lemma is a variation on the classical proof of Ruelle's inequality.

We are able to analyze the dynamics of entropy-expanding maps with respect to large entropy measures rather completely.

Theorem 9. *Let $f : M \to M$ be a C^∞ self-map of a compact manifold. Assume that f is entropy-expanding. Then:*

- *f has finitely many maximum measures;*
- *its periodic points satisfies a multiplicative lower bound.*

This can be understood as generalization of the Markov property which corresponds to partition having boundaries with essentially finite forward or backward orbits. The proof of the theorem involves a partition whose boundaries are pieces of smooth submanifolds, therefore of entropy bounded by $H^{d-1}_{\text{top}}(f)$.

In [9], we are able to define a nice class of symbolic systems, called *puzzles of quasi-finite type*, which contains the suitably defined symbolic representations of entropy-expanding maps satisfying a technical condition and have the above properties. Moreover, their periodic points define zeta functions with meromorphic extensions and their natural extensions can be classified up to entropy-conjugacy in the same way as interval maps.

6.3. **Entropy-Hyperbolicity.** Entropy-expanding maps are *never diffeomorphisms*. Indeed, they have ergodic invariant measures which have nonzero entropy and only positive Lyapunov exponents. Wrt the inverse diffeomorphism these measures have the same nonzero entropy but only negative Lyapunov exponents, contradicting Ruelle's inequality. Thus we need a different notion for diffeomorphism.

Definition 33. *The* **unstable (entropy) dimension** *is:*

$$d_u(f) := \min\{0 \leq k \leq d : H^k_{\text{top}}(f) = h_{\text{top}}(f)\}.$$

If f is a diffeomorphism, then the **stable dimension** *is: $d_s(f) := d_u(f^{-1})$ (if f not a diffeomorphism we set $d_s(f) = 0$).*

Observe that f is entropy-expanding if and only if $d_u(f)$ coincides with the dimension of the manifold.

Lemma 34. *Let $f : M \to M$ be a C^r self-map of a compact d-dimensional manifold with $r > 1$. Then:*

$$d_u(f) + d_s(f) \leq d.$$

Proof. Theorem 8 implies that measures with entropy $> H^{d_u(f)-1}_{\text{top}}(f)$ have at least $d_u(f)$ positive exponents. The same reasoning applied to f^{-1} shows that such measures have at least $d_s(f)$ negative exponents. By the variational principle such measures exist. Hence $d_u(f) + d_s(f) \leq d$. □

We can now propose our definition:

Definition 35. *A diffeomorphism such that* $d_u(f)+d_s(f) = d$ *is* **entropy-hyperbolic**.

Obviously surface diffeomorphisms with non-zero topological entropy are entropy-hyperbolic.

Exactly as above, we obtain from Theorems 8 and 4:

Theorem 10. *Let* $f : M \to M$ *be a* C^r *diffeomorphism of some compact manifold with* $r > 1$. *Assume that* f *is entropy-hyperbolic. Then:*

- *all ergodic invariant probability measures with entropy close enough to the topological entropy have the absolute value of their Lyapunov exponents bounded away from zero;*
- *their periodic points satisfy a logarithmic lower bound;*
- *they contain horseshoes with topological entropy arbitrarily close to that of* f.

Corollary 36. *The set of entropy-hyperbolic diffeomorphisms of a compact manifold is open in the* C^∞ *topology.*

6.4. **Examples of Entropy-Hyperbolic Diffeomorphisms.** The techniques of [6] yield:

Lemma 37. *Linear toral automorphisms are entropy-hyperbolic if and only if they are hyperbolic in the usual sense: no eigenvalue lies on the unit circle.*

The following condition is easily seen to imply entropy-hyperbolicity:

Lemma 38. *Let* $f : M \to M$ *be a diffeomorphism of a compact manifold of dimension* d. *Assume that there are two integers* $d_1 + d_2 = d$ *such that:*

$$\log \|\Lambda^{d_1-1} Tf\| < h_{\rm top}(f) \text{ and } \log \|\Lambda^{d_2-1} Tf\| < h_{\rm top}(f).$$

Then f *is entropy-hyperbolic.*

7. Further directions and Questions

We discuss some developping directions and ask some questions.

7.1. **Variational Principles.** It seems reasonable to conjecture the following *topological variational principle* for dimensional entropies, at least for C^∞ self-maps and diffeomorphisms:

In each dimension, there is a C^∞ disk with maximum topological entropy, i.e., $h^k_{\rm top}(f)$.

Does it fail for finite smoothness?

A probably more interesting but more delicate direction would be an *ergodic variational principle*. Even its formulation is not completely clear. A possibility would be as follows:

For each dimension k, $h^k_{\rm top}(f)$ is the supremum of the entropies of k-disks contained in unstable manifolds of points in any set of full measure with respect to all invariant probability measures.

7.2. **Dimensional Entropies of Examples.** Let $f_i : M_i \to M_i$ are smooth maps for $i = 1, \dots, n$ and consider the following formula:

$$H^k_{\mathrm{top}}(f_1 \times \cdots \times f_n) = \max_{\ell_1+\cdots+\ell_n=k} H^{\ell_1}_{\mathrm{top}}(f) + \cdots + H^{\ell_n}_{\mathrm{top}}(f).$$

This is only known in special cases –see [3]. It would imply that product of entropy-expanding maps are again entropy-expanding.

If $f : M \to M$ is an expanding map of a compact manifold, is it true that $H^{d-1}_{\mathrm{top}}(f) < h_{\mathrm{top}}(f)$. Note that this fails for piecewise expanding maps (think of a limit set containing an isolated invariant curve with maximum entropy).

Likewise is an Anosov diffeomorphism, even far from linear, entropy-hyperbolic?

Find examples where $h^{k,C^r}_{\mathrm{top}}(f) < H^{k,C^r}_{\mathrm{top}}(f)$.

7.3. **Other types of dimensional complexity.** Other "dimensional complexities" have been investigated from growth rates of multi(co)vectors for the Kozlovski entropy formula [21] to the currents which are fundamental to multidimensional complex dynamics [13] and the references therein.

How do they relate to the above dimensional entropies?

7.4. **Necessity of Topological Assumptions.** We have seen in Sect. 2.1 that, for maps, the assumption of no zero Lyapunov exponent for the large entropy measure, (or even that these exponents are bounded away from zero) is not enough for our purposes (e.g., finiteness of the number of maximum measures). Such results seem to require more uniform assumptions, like the one we make on dimensional entropies.

Is it the same for diffeomorphisms? That is, can one find diffeomorphisms with infinitely many maximum measures, all with exponents bounded away from zero?

Let f be a C^r self-map of a compact manifold. Assume that there are numbers $h < h_{\mathrm{top}}(f)$, $\lambda > 0$, such that the Lyapunov exponents of any ergodic invariant measure with entropy at least h fall outside of $[-\lambda, \lambda]$. Assume also that the set of invariant probability measures with entropy $\geq h$ is compact. Does it follow that there are only finitely many maximum measures?

7.5. **Entropy-Hyperbolicity.** In a work in progress with T. Fisher, we show that the condition of Lemma 38 is satisfied by a version of a well-known example of robustly transitive, non-uniformly hyperbolic diffeomorphism of $\mathbb{T}^4$ due to Bonatti and Viana [1]. Building nice center-stable and center-unstable invariant laminations, we expect be able to show the same properties as in Theorem 9.

I however conjecture that the finite number of maximum measures, etc. should in fact hold for every C^∞ entropy-expanding diffeomorphism, even when there is no such nice laminations. Of course this contains the case of surface diffeomorphisms which is still open (see Conjecture 1), despite the result on a toy model [8].

7.6. **Generalized Entropy-Hyperbolicity.** It would be interesting to have a *more general notion of entropy-hyperbolicity.* For instance, if a hyperbolic toral automorphism is entropy-hyperbolic, this is not the case for the disjoint union of two such systems of the same dimenion if they have distinct stable dimensions. It may be possible to "localize" the definition either near points or near invariant measures to avoid these stupid obstructions (this is one motivation for the above question on variational principles for dimensional entropies).

If one could remove such obstructions, the remaining ones could reflect basic dynamical phenomena opening the door to a speculative "entropic Palis program".

Appendix A. C^r sizes

We explain how to measure the C^r size of singular disks of a compact manifold M of dimension d. Here $1 \leq r \leq \infty$.

We select a finite atlas $\mathcal{A}$ made of charts $\chi_i : U_i \subset \mathbb{R}^d \to M$ such that changes of coordinates $\chi_i^{-1} \circ \chi_j$ are C^r-diffeomorphisms of open subsets of $\mathbb{R}^d$. Then we define, for any singular k-disk ϕ:

$$\|\phi\|_{C^r} := \sup_{x \in M} \inf_{U_i \ni x} \max_{s_1+\dots+s_k \leq r} \max_{1 \leq j \leq d} |\partial_{t_1}^{s_1} \dots \partial_{t_k}^{s_k} (\pi_j \circ \chi_i \circ \phi)(t_1, \dots, t_k)|$$

where the above partial derivatives are computed at $t = \chi_i^{-1}(x)$ and $\pi_j(u_1, \dots, u_d) = u_j$.

Fact 39. *If $\|\cdot\|_{C^r}$ and $\|\cdot\|'_{C^r}$ are two C^r size defined by the above procedure, there exists a constant K such that, for any C^r k-disk $\phi : Q^k \to M$:*

$$\|\phi\|_{C^r} \leq K \cdot \|\phi\|'_{C^r}.$$

Fact 40. *Let $\phi : Q^k \to M$ be a C^r disk for some finite $r \geq 1$. Then, for any $t_0 \in Q^k$, there exists a C^∞ approximation $\phi_\infty : Q^k \to M$ such that:*

$$\forall t \in Q^k \quad d(\phi_\infty(t), \phi(t)) \leq \|\phi\|_{C^r} \|t - t_0\|^r.$$

and $\|\phi_\infty\|_{C^\infty} \leq 2\|\phi\|_{C^r}$.

This is easily shown by considering a neighborhood of $\phi(t_0)$ contained in a single chart of $\mathcal{A}$ and approximating ϕ by its Taylor expansion in that chart.

References

[1] C. Bonatti, L. Diaz, M. Viana, *Dynamics beyond uniform hyperbolicity,* Springer-Verlag, Berlin, 2005.

[2] M. Boyle, J. Buzzi, R. Gomez, *Almost isomorphism for countable state Markov shifts,* J. Reine Angew. Math. 592 (2006), 23–47.

[3] D. Burguet, *Entropie des systèmes dynamiques différentiables,* thèse en préparation.

[4] J. Buzzi, *Représentation markovienne des applications régulières de l'intervalle,* Thèse de l'Université Paris-Sud, 1995.

[5] J. Buzzi, *Intrinsic ergodicity of smooth interval maps,* Israel J. Math. 100 (1997), 125–161.

[6] J. Buzzi, *Ergodicité intrinsèque de produits fibrés d'applications chaotiques unidimensionelles,* Bull. Soc. Math. France 126 (1998), no. 1, 51–77.

[7] J. Buzzi, *On entropy-expanding maps,* preprint 2000.

[8] J. Buzzi, *Measures of Maximum Entropy for Piecewise Affine Surface Homeomorphisms,* Ergod. th. dynam. syst. (to appear).

[9] J. Buzzi, *Puzzles of Quasi-Finite Type, Symbolic Dynamics and Zeta Functions,* Ann. Inst. Fourier (to appear).

[10] J. Buzzi, *Cours de Saint-Flour,* in preparation.

[11] K. Cogswell, *Entropy and volume growth,* Ergodic Theory Dynam. Systems 20 (2000), 77–84.

[12] M. Gromov, *Entropy, homology and semialgebraic geometry,* Séminaire Bourbaki, Vol. 1985/86. Astérisque No. 145-146 (1987), 5, 225–240.

[13] M. Gromov, *On the entropy of holomorphic maps,* Enseign. Math. (2) 49 (2003), 217–235.

[14] B. M. Gurevich, *Shift entropy and Markov measures in the path space of a denumerable graph (Russian),* Dokl. Akad. Nauk SSSR 192 (1970), 963–965; English transl.: Sov. Math. Dokl. 11 (1970), 744–747.

[15] F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy II,* Israel J. Math. 38 (1981), 107–115.

[16] F. Hofbauer, *Periodic points for piecewise monotonic transformations,* Ergodic Theory Dynam. Systems 5 (1985), 237–256.
[17] V. Kaloshin, *Generic diffeomorphisms with superexponential growth of number of periodic orbits,* Comm. Math. Phys. 211 (2000), 253–271.
[18] V. Kaloshin, B. Hunt, *Stretched exponential estimates on growth of the number of periodic points for prevalent diffeomorphisms. I,* Ann. of Math. (2) 165 (2007), 89–170.
[19] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms,* Inst. Hautes Etudes Sci. Publ. Math. No. 51 (1980), 137–173.
[20] A. Katok, B. Hasselblatt, *Introduction to the modern theory of dynamical systems,* Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995.
[21] O. Kozlovski, *An integral formula for topological entropy of C^∞ maps* Ergodic Theory Dynam. Systems 18 (1998), no. 2, 405–424.
[22] M. Misiurewicz, *Diffeomorphism without any measure with maximal entropy,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 21 (1973), 903–910.
[23] S. Newhouse, *Entropy and volume,* Ergodic Theory Dynam. Systems 8* (1988), Charles Conley Memorial Issue, 283–299.
[24] K. Oliveira, M. Viana, *Existence and uniqueness of maximizing measures for robust classes of local diffeomorphisms,* Discrete Contin. Dyn. Syst. 15 (2006), no. 1, 225–236.
[25] K. Oliveira, M. Viana, *Thermodynamical formalism for robust classes of potentials and non-uniformly hyperbolic maps,* Ergodic Theory Dynam. Systems 28 (2008), no. 2, 501–533.
[26] D. Vere-Jones, *Ergodic properties of non-negative matrices,* Pac. J. Math. 22 (1967), 361–386.
[27] Y. Yomdin, *Volume growth and entropy,* Israel J. Math. 57 (1987), no. 3, 285–300.